6.3 Signal Space Concepts

As in the case of vectors, we now develop a parallel treatment for a set of signals.

Definition 6.34.

(a) The **inner product** of two real-valued signals $x_1(t)$ and $x_2(t)$ is denoted by $\langle x_1(t), x_2(t)\rangle$ and defined by

$$\langle x_1, x_2\rangle \equiv \langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2(t)dt.$$

(b) The signals are **orthogonal** if their inner product is zero.

(c) The **norm** of a signal is defined as

$$\|x(t)\| = \sqrt{\langle x(t), x(t)\rangle} = \sqrt{E_x}$$

where E_x is the energy in $x(t)$:

$$\|x(t)\|^2 = \langle x(t), x(t)\rangle = \int_{-\infty}^{\infty} x(t)x(t)dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \equiv E_x$$

(d) A collection of N signals is **orthonormal** if the signals are (1) orthogonal and their (2) norms are all unity.

Example 6.35. Consider the two waveforms shown in Figure 16.

$$E_{s_1} = \int_{-\infty}^{\infty} s_1^2(t)dt = \int_0^2 1\, dt = 2$$

$$E_{s_2} = \int_{-\infty}^{\infty} s_2^2(t)dt = \int_0^2 1\, dt = 2$$

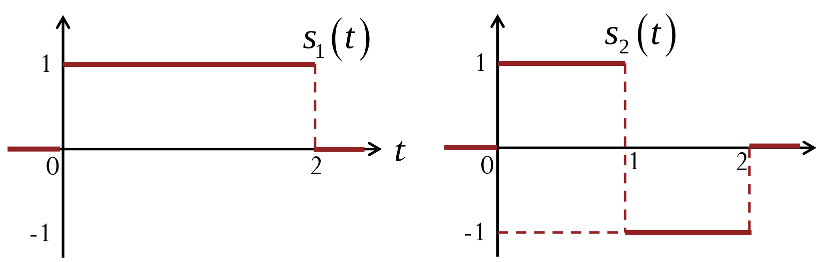
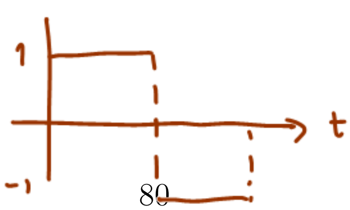


Figure 16: Two Waveforms in Example 6.35

$$\|s_1\| = \sqrt{E_{s_1}} = \sqrt{2}$$

$$\|s_2\| = \sqrt{2}$$



$$\langle s_1, s_2\rangle = \int_{-\infty}^{\infty} \underbrace{s_1(t)s_2(t)}\, dt = 0 \Rightarrow s_1 \text{ and } s_2 \text{ are orthogonal}$$

Definition 6.36.

(orthogonal)

(a) The **projection** of $x_2(t)$ to $x_1(t)$ is given by

$$\text{proj}_{x_1} x_2 = \text{proj}_{x_1(t)} x_2(t) = \frac{\langle x_2, x_1\rangle}{\langle x_1, x_1\rangle} x_1 = \frac{\langle x_2(t), x_1(t)\rangle}{\langle x_1(t), x_1(t)\rangle} x_1(t) = \frac{\langle x_2(t), x_1(t)\rangle}{E_{x_1}} x_1(t)$$

(b) The **cross-correlation coefficient** of $x_1(t)$ and $x_2(t)$ is defined as

$$\rho_{x_1,x_2} = \frac{\langle x_1(t), x_2(t)\rangle}{\sqrt{E_{x_1} E_{x_2}}}.$$

- $\text{proj}_{x_1(t)} x_2(t) = \sqrt{E_{x_2}} \rho_{x_2,x_1} \frac{x_1(t)}{\sqrt{E_{x_1}}}$

Example 6.37. For the two waveforms shown in Figure 16,

$$\text{proj}_{s_1} s_2 = \frac{\langle s_2, s_1\rangle}{\langle s_1, s_1\rangle} s_1 = \frac{0}{2} s_1 \equiv 0$$

This is not just a number 0. This is a function of time whose value is 0 at all time.

6.38. Similar to 6.30, the **Gram-Schmidt Orthogonalization Procedure (GSOP)** can be used to construct a set of orthonormal waveforms from a set of finite energy signal waveforms: $\{s_j(t), j = 1, 2, \dots, M\}$.

Same as the vector version.

The first orthonormal function is simply constructed as

$$\phi_1(t) = \frac{u_1(t)}{\sqrt{E_{u_1}}} = \frac{s_1(t)}{\sqrt{E_{s_1}}}.$$

The subsequent orthonormal functions are found as follows:

$$\phi_i(t) = \frac{u_i(t)}{\sqrt{E_{u_i}}},$$

where the unnormalized basis function $u_i(t)$ is given by

$$u_i(t) = s_i(t) - \sum_{k=1}^{i-1} \text{proj}_{u_k(t)} s_i(t).$$

and

$$\text{proj}_{u_k(t)} s_i(t) = \frac{\langle s_i(t), u_k(t)\rangle}{\langle u_k(t), u_k(t)\rangle} u_k(t) = \langle s_i(t), \phi_k(t)\rangle \phi_k(t)$$

As with the GSOP for vectors, we also discard the zero functions. In general, the final number of orthonormal functions, N, is less than or equal to the number of given waveforms, M, depending on one of the two possibilities:

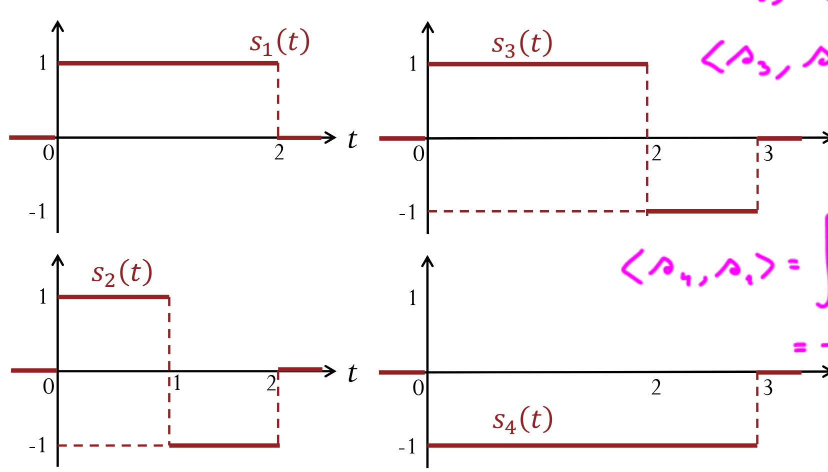
(a) If the waveforms $\{s_j(t), j = 1, 2, \ldots, M\}$ form a linearly independent set, then $N = M$.

(b) If the waveforms $\{s_j(t), j = 1, 2, \ldots, M\}$ are not linearly independent, then $N < M$.

Example 6.39. Consider the four waveforms illustrated in Figure 17. Use the Gram-Schmidt orthogonalization procedure (where the waveforms are applied **in the order given**) to find the orthonormal basis waveforms $\phi_1(t)$, $\phi_2(t)$, ... whose linear combinations can be used to represent the four waveforms.

$\langle s_3, s_1 \rangle = 2$

$\langle s_3, s_2 \rangle = 0$



$\langle s_4, s_1 \rangle = \int_{-1}^{\ldots} \ldots \, dt$

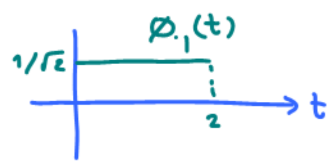
$= -2$

Figure 17: Four signals for orthogonalization in Example 6.39

"direction of the first axis"

- First we set $u_1(t) = s_1(t)$.
  $E_{u_1} = E_{s_1} = 2$
  $\phi_1(t) = \frac{u_1(t)}{\sqrt{E_{u_1}}} = \frac{s_1(t)}{\sqrt{E_{s_1}}} = \frac{1}{\sqrt{2}} s_1(t)$

the first axis



$\Rightarrow s_1(t) = \sqrt{2}\, \phi_1(t)$

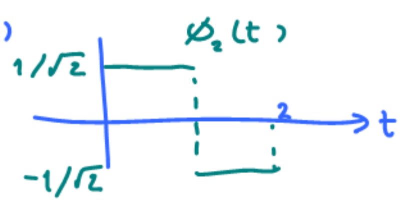
- $u_2(t) = s_2(t) - \underbrace{\mathrm{proj}_{u_1} s_2}_{= \mathrm{proj}_{s_1} s_2 = 0} = s_2(t)$

direction of the second axis

Ex. 6.35

Ex. 6.37

$E_{u_2} = E_{s_2} = 2$
$\phi_2(t) = \frac{u_2(t)}{\sqrt{E_{u_2}}} = \frac{1}{\sqrt{2}} s_2(t)$



$\Downarrow$

$s_2(t) = \sqrt{2}\, \phi_2(t)$

$u_3 = s_3 - s_1$

$s_3 = s_1 + u_3 = u_1 + u_3 = \sqrt{E_{u_1}}\,\phi_1 + \sqrt{E_{u_3}}\,\phi_3$

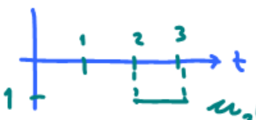
$= \sqrt{2}\,\phi_1 + 1\,\phi_3$

- $u_3(t) = s_3(t) - \underbrace{\text{proj}_{u_1} s_3}_{\substack{= \text{proj}_{s_1} s_3 \\ = \frac{\langle s_3, s_1 \rangle}{E_{s_1}} s_1 \\ = \frac{2}{2} s_1 = s_1}} - \underbrace{\text{proj}_{u_2} s_3}_{\substack{= \text{proj}_{s_2} s_3 \\ = \frac{\langle s_3, s_2 \rangle}{E_{s_2}} s_2 \\ = \frac{0}{2} s_2 = 0}}$

  $= s_3 - s_1 - 0$

  $E_{u_3} = 1$

  $\phi_3(t) = \frac{u_3(t)}{\sqrt{E_{u_3}}} = \frac{u_3(t)}{1} = u_3(t)$



- $u_4(t) = s_4(t) - \underbrace{\text{proj}_{u_1} s_4}_{\substack{= \text{proj}_{s_1} s_4 \\ = \frac{\langle s_4, s_1 \rangle}{E_{s_1}} s_1 \\ = \frac{-2}{2} s_1 = -s_1}} - \underbrace{\text{proj}_{u_2} s_4}_{= 0} - \cancel{\text{proj}_{u_1}} s_4 - \underbrace{\text{proj}_{u_3} s_4}_{= u_3}$

  discarded (not in the basis; not a new axis)

  $= s_4 + s_1 - u_3 = 0$

$s_4 = -s_1 + u_3 = -u_1 + u_3$

$= -\sqrt{E_{u_1}}\,\phi_1 + \sqrt{E_{u_3}}\,\phi_3 = -\sqrt{2}\,\phi_1 + \phi_3$

6.40. Once we have constructed[15] the set of, say N, orthonormal waveforms $\{\phi_i(t), i = 1, 2, \ldots, N\}$, we can express the signals $s_i(t)$ as linear combinations of the N orthonormal basis functions $\phi_i(t)$. Thus, we may write

$$s_j(t) = \sum_{i=1}^{N} s_i^{(j)} \phi_i(t) \tag{34}$$

where the constants (weights)

$$s_i^{(j)} = \langle s_j(t), \phi_i(t) \rangle . \tag{35}$$

[15] We have shown how this set can be constructed from GSOP. However, in practice, this set may be derived from different procedure.

Note that $s_i^{(j)}\phi_i(t) = \langle s_j(t), \phi_i(t)\rangle\, \phi_i(t)$ can be geometrically interpreted as the projection of the signal $s_j(t)$ onto the ith axis, $\phi_i(t)$.

Based on (34), each signal may be represented by the vector (or sequence)

$$\mathbf{s}^{(j)} = (s_1^{(j)}, s_2^{(j)}, \ldots, s_N^{(j)})^T, \tag{36}$$

or, equivalently, as a point in the N-dimensional (in general, complex) signal space.

The (mathematical/conceptual) conversion/mapping from waveform to it corresponding vector in (36) and (35) is shown in Figure 18a. The inverse mapping from vector to waveform in (34) is shown in Figure 18b.

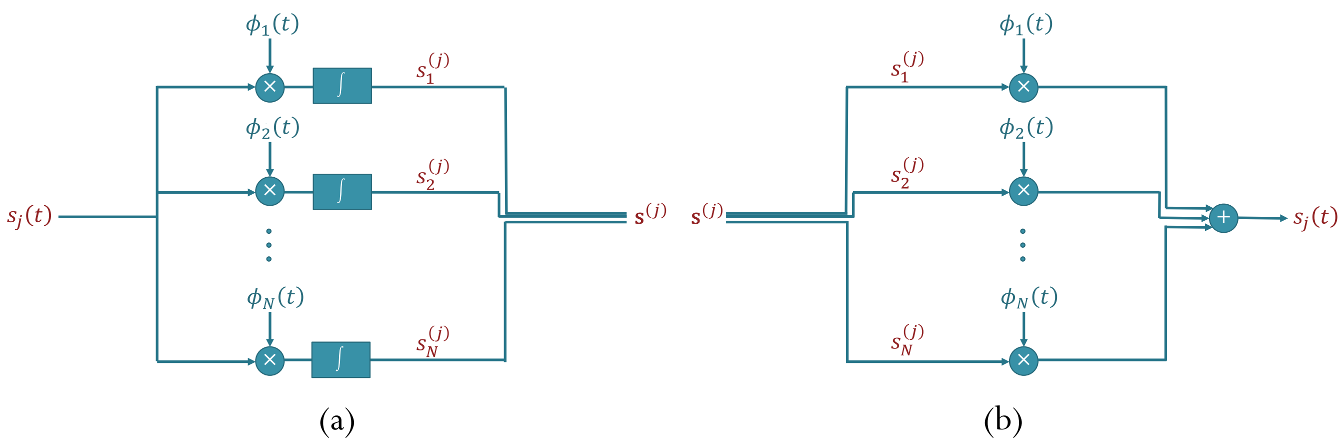


Figure 18: Waveform to vector (a), and vector to waveform (b) mappings.

Example 6.41. For the four waveforms in Example 6.39 and the orthonormal basis derived from GSOP,

$$s_1(t) = \sqrt{2}\,\phi_1(t) + 0\,\phi_2(t) + 0\,\phi_3(t) \Rightarrow \vec{s}^{(1)} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

$$s_2(t) = 0\,\phi_1(t) + \sqrt{2}\,\phi_2(t) + 0\,\phi_3(t) \Rightarrow \vec{s}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

$$s_3(t) = \sqrt{2}\,\phi_1(t) + 0\,\phi_2(t) + 1\,\phi_3(t) \Rightarrow \vec{s}^{(3)} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}$$

$$s_4(t) = -\sqrt{2}\,\phi_1(t) + 0\,\phi_2(t) + 1\,\phi_3(t) \Rightarrow \vec{s}^{(4)} = \begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$$

Definition 6.42. From 6.40, a set of M signals $\{s_j(t), j = 1, 2, \ldots, M\}$ can be represented by a set of M vectors $\{\mathbf{s}^{(j)}\}$ in the N-dimensional space. The corresponding set of vectors is called the **signal space representation**, or **constellation**, of $\{s_j(t), j = 1, 2, \ldots, M\}$.

6.43. From the orthonormality of the basis, we have

(a) the inner product of two signals is equal to the inner product of the corresponding vectors:

$$\langle s_i(t), s_j(t) \rangle = \left\langle \mathbf{s}^{(i)}, \mathbf{s}^{(j)} \right\rangle.$$

Ex. 6.35

Ex. $\langle s_1, s_2 \rangle = 0$

$\langle \vec{s}^{(1)}, \vec{s}^{(2)} \rangle = 0 + 0 + 0 = 0$

(b) $E_j \equiv E_{s^{(j)}} = \|s_j(t)\|^2 = \left\|\mathbf{s}^{(j)}\right\|^2$.

Ex $E_{s_1} = \|\vec{s}^{(1)}\|^2 = (\sqrt{2})^2 + 0^2 + 0^2 = 2$

6.44. It should be emphasized, however, that the functions $\{\phi_i(t)\}$ obtained from the Gram-Schmidt procedure are not unique. If we alter the order in which the orthogonalization of the signals $\{s_j(t)\}$ is performed, the orthonormal waveforms will be different and the corresponding vector representation of the signals $\{s_j(t)\}$ will depend on the resulting orthonormal functions $\{\phi_i(t)\}$. Nevertheless, the dimensionality of the signal space (N) will not change, and the vectors $\mathbf{s}^{(j)}$ will retain their geometric configuration; i.e., their lengths and their inner products will be invariant to the choice of the orthonormal functions $\{\phi_i(t)\}$.